Score sets in oriented bipartite graphs

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Abstract. The set A of distinct scores of the vertices of an oriented bipartite graph D(U, V) is called its score set. We consider the following question: given a finite, nonempty set A of positive integers, is there an oriented bipartite graph D(U, V) such that score set of D(U, V) is A? We conjecture that there is an affirmative answer, and verify this conjecture when $| A | = 1$, 2, 3, or when A is a geometric or arithmetic progression.

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1. Introduction. An oriented graph is a digraph with no symmetric pairs of directed arcs and without loops. Let D be an oriented graph with vertex set V = $\{v_1, v_2, \ . \ . \ . \ , v_n\}$, and let d_v^+ and d_v^- denote the outdegree and indegree respectively of a vertex v. Avery [1] defined $a_v = n-1+d_v^+ - d_v^-$, the score of v, so that $0 \le a_v \le 2n-2$. Then, the sequence $[a_1, a_2, \ . \ . \ . \ , a_n]$ in non-decreasing order is called the score sequence of D.

Avery [1] obtained the following criterion for score sequences in oriented graphs .

Theorem 1.1. A non-decreasing sequence of non-negative integers $[a_1, a_2, \ . \ . \ . \ , a_n]$ is the score sequence of an oriented graph if and only if

$$\sum_{i=1}^{k} a_i \ge \ k(k-1), \quad \text{for } 1 \le k \le n,$$

with equality when k = n.

Pirzada and Naikoo [7] obtained the following results for score sets in oriented graphs.

Theorem 1.2. Let A = $\{a, ad, ad^2, \ . \ . \ . \ , ad^n\}$, where a and d are positive integers with a $>$ 0 and d $>$ 1. Then, there exists an oriented graph D with score set A, except for a = 1, d = 2, n $>$ 0 and for a = 1, d = 3, n $>$ 0.

Theorem 1.3. If $a_1, a_2, \ . \ . \ . \ , a_n$ are n non-negative integers with $a_1 < a_2 < \ . \ . \ . \ < a_n$, then there exists an oriented graph D with score set A = $\{a_1', a'_2, \ . \ . \ . \ , a'_n\}$, where

$$a_i' = \begin{cases} a_{i-1}+a_i+1, & for\ i > 1, \\ a_i, & for\ i = 1. \end{cases}$$

The study of score sets in tournaments (complete oriented graphs) can be found in [2, 5, 8, 10, 11].

An oriented bipartite graph is the result of assigning a direction to each edge of a simple bipartite graph. Let U = $\{u_1, u_2, \ . \ . \ . \ , u_m\}$ and V = $\{v_1, v_2, \ . \ . \ . \ , v_n\}$ be the parts of an oriented bipartite graph D(U, V). For any vertex x in D(U, V) , let d_x^+ and d_x^- respectively be the outdegree and indegree of x. Define $a_u = n + d_u^+ - d_u^-$ and $b_v = m + d_v^+ - d_v^-$ respectively as the scores of u in U and v in V. Clearly , $0 \le a_u \le 2n$ and $0 \le b_v \le 2m$. The sequences $[a_1, a_2, \ . \ . \ . \ , a_m]$ *and* $[b_1, b_2, \ . \ . \ . \ , b_n]$ in non-decreasing order are called the score sequences of D(U, V).

The following result due to Pirzada, Merajuddin and Yin [4] is the bipartite version of Theorem 1.1.

Theorem 1.4. Two non-decreasing sequences $[a_1, a_2, \ . \ . \ . \ , a_m]$ *and* $[b_1, b_2, \ . \ . \ . \ , b_n]$ of non-negative integers are the score sequences of some oriented bipartite graph if and only if

$$\sum_{i=1}^{p} a_i + \sum_{j=1}^{q} b_j \ge 2pq, \ \text{ for } 1 \le p \le m \text{ and } 1 \le q \le n,$$

with equality when p = m and q = n.

The study of score sets for bipartite tournaments (complete oriented bipartite graphs) can be found in [3, 9, 12] and for k-partite tournaments (complete oriented k-partite graphs) in [6].

2. Score sets in oriented bipartite graphs

Definition. The set A of distinct scores of the vertices in an oriented bipartite graph D(U, V) is called its score set. If there is an arc from a vertex u to a vertex v, then we say that vertex u dominates vertex v.

We have the following results.

Theorem 2.1. Every singleton or doubleton set of positive integers is a score set of some oriented bipartite graph.

Proof. **Case I.** Let A = {a}, where a is a positive integer. When a is even, construct an oriented bipartite graph D(U, V) as follows.

Let

$$\mathrm{U} = X_1 \cup X_2, \quad \mathrm{V} = Y_1 \cup Y_2$$

with $X_1 \cap X_2 = \phi, Y_1 \cap Y_2 = \phi, |X_1| = |X_2| = |Y_1| = |Y_2| = \frac{a}{2}$. Let every vertex of X_i dominates each vertex of Y_i, and every vertex of Y_i dominates each vertex of X_j whenever $i \neq j$ so that we get the oriented bipartite graph D(U, V) with

$$|U| = |X_1| + |X_2| = |Y_1| + |Y_2| = |V| = \frac{a}{2} + \frac{a}{2} = \mathrm{a},$$

and the scores of vertices

$a_{x_1} = |V| + |Y_1| - |Y_2| = |U| + |X_1| - |X_2| = a_{y_2} = a + \frac{a}{2} - \frac{a}{2} = a$, for all $x_1 \in X_1, y_2 \in Y_2$

and $a_{x_2} = |V| + |Y_2| - |Y_1| = |U| + |X_2| - |X_1| = a_{y_1} = a + \frac{a}{2} - \frac{a}{2} = a$, for all $x_2 \in X_2, y_1 \in Y_1$.

Therefore, score set of D(U, V) is A = {a}.

Now, when a is odd, construct an oriented bipartite graph D(U, V) as follows.

Let

$$U = X_1 \cup X_2 \cup \{x\}, \quad V = Y_1 \cup Y_2 \cup \{y\}$$

with $X_1 \cap X_2 = \phi, X_i \cap \{x\} = \phi, Y_1 \cap Y_2 = \phi, Y_i \cap \{y\} = \phi, |X_1| = |X_2| = |Y_1| = |Y_2| = \frac{a-1}{2}$. Let every vertex of X_i dominates each vertex of Y_i, and every vertex of Y_i dominates each vertex of X_j whenever i $\neq$ j so that we get the oriented bipartite graph D(U, V) with

$$|U| = |X_1| + |X_2| + |\{x\}| = |Y_1| + |Y_2| + |\{y\}| = |V| = \frac{a-1}{2} + \frac{a-1}{2} + 1 = a,$$

and the scores of vertices

$a_{x_1} = |V| + |Y_1| - |Y_2| = |U| + |X_1| - |X_2| = a_{y_2} = a + \frac{a-1}{2} - \frac{a-1}{2} = a$, for all $x_1 \in X_1, y_2 \in Y_2$,

$a_{x_2} = |V|+|Y_2|-|Y_1| = |U|+|X_2|-|X_1| = a_{y_1} = a+\frac{a-1}{2}-\frac{a-1}{2} = a$, for all $x_2 \in X_2, y_1 \in Y_1$ and $a_x = |V| + 0 - 0 = |U| + 0 - 0 = a_y = a$, for the vertices x and y.
Thus, score set of D(U, V) is A = {a}.

Note that an empty oriented bipartite graph D(U, V) with $|U| = |V| = a$ has also score set A = {a}.

Case II. Let $A = \{a_1, a_2\}$, where a_1 and a_2 are positive integers with $a_1 < a_2$. As in case I, there exists an oriented bipartite graph D(U, V) with $|U| = |V| = a_1$, and the scores of vertices $a_u = a_v = a_1$, for all $u \in U, v \in V$.

Since $a_2 > a_1$ or $a_2 - a_1 > 0$, construct oriented bipartite graph $D(U_1, V_1)$ as follows.

Let $U_1 = U \cup X, V_1 = V, U \cap X = \phi, |X| = a_2 - a_1$. Let there be no arc between the vertices of V and X, so that we get the oriented bipartite graph D(U_1, V_1) with

$$|U_1| = |U| + |X| = a_1 + a_2 - a_1 = a_2, |V_1| = a_1,$$

and the scores of vertices

$a_u = a_1$, for all $u \in U$,

$a_x = |V_1| + 0 - 0 = a_1$, for all $x \in X$,

and $a_v = a_1 + |X| = a_1 + a_2 - a_1 = a_2$, for all $v \in V$.
Hence, score set of $D(U_1, V_1)$ is A =$\{a_1, a_2\}$.

Again, note that an empty oriented bipartite graph D(U, V) with $|U| = a_1, |V| = a_2$ has also score set $A = \{a_1, a_2\}$.

Theorem 2.2. Every set of three positive integers is a score set of some oriented bipartite graph.
Proof. Let $A = \{a_1, a_2, a_3\}$, where a_1, a_2, a_3 are positive integers with $a_1 < a_2 < a_3$.

First assume $a_3 > 2a_2$ so that $a_3 - 2a_2 > 0$, and since $a_2 > a_1$, therefore $a_3 - 2a_1 > 0$. Now, construct an oriented bipartite graph D(U, V) as follows.
Let $U = X_1 \cup X_2, V = Y_1 \cup Y_2$ with $X_1 \cap X_2 = \phi, Y_1 \cap Y_2 = \phi$, $|X_1| = a_2, |X_2| = a_3 - 2a_2, |Y_1| = a_1, |Y_2| = a_3 - 2a_1$. Let every vertex of X_2 dominates each vertex of Y_1, and every vertex of Y_2 dominates each vertex of X_1, so that we get the oriented bipartite graph D(U, V) with

$$|U| = |X_1| + |X_2| = a_2 + a_3 - 2a_2 = a_3 - a_2,$$
$$|V| = |Y_1| + |Y_2| = a_1 + a_3 - 2a_1 = a_3 - a_1,$$

and the scores of vertices

$a_{x_1} =| V | +0 - (a_3 - 2a_1) = a_3 - a_1 - a_3 + 2a_1 = a_1$, for all $x_1 \in X_1$,

$a_{x_2} =| V | +a_1 - 0 = a_3 - a_1 + a_1 = a_3$, for all $x_2 \in X_2$,

$a_{y_1} =| U | +0 - (a_3 - 2a_2) = a_3 - a_2 - a_3 + 2a_2 = a_2$, for all $y_1 \in Y_1$,

and $a_{y_2} =| U | +a_2 - 0 = a_3 - a_2 + a_2 = a_3$, for all $y_2 \in Y_2$.
Therefore, score set of D(U, V) is $A = \{a_1, a_2, a_3\}$.

Now, assume $a_3 \leq 2a_2$ so that $2a_2 - a_3 \geq 0$. Construct an oriented bipartite graph D(U, V) as follows.

Let $U = X_1, V = Y_1 \cup Y_2$ with $Y_1 \cap Y_2 = \phi, |X_1| = a_2, |Y_1| = a_1, |Y_2| = a_2 - a_1$. Let every vertex of Y_2 dominates $a_3 - a_2$ vertices of X_1 (out of a_2), so that we get the oriented bipartite

graph D(U, V) with

$$|U| = |X_1| = a_2, |V| = |Y_1| + |Y_2| = a_1 + a_2 - a_1 = a_2,$$

and the scores of vertices

$a_{x_1} =\mid V \mid +0 - (a_2 - a_1) = a_2 - a_2 + a_1 = a_1,\quad$ for the $a_3 - a_2$ vertices of X_1,

$a_{x_1'} =\mid V \mid +0 - 0 = a_2$, for the remaining $a_2 - (a_3 - a_2) = 2a_2 - a_3$ vertices of X_1,

$a_{y_1} =\mid U \mid +0 - 0 = a_2,\quad$ for all $y_1 \in Y_1$,

and $a_{y_2} =\mid U \mid +a_3 - a_2 - 0 = a_2 + a_3 - a_2 = a_3,\quad$ for all $y_2 \in Y_2$.

Thus, score set of D(U, V) is $A = \{a_1, a_2, a_3\}$.

The next result shows that every set of positive integers in geometric progression is a score set of some oriented bipartite graph.

Theorem 2.3. Let $A = \{a, ad, ad^2, \ . \ . \ . \ , ad^n\}$, where a and d are positive integers with $a > 0$ and $d > 1$. Then, there exists an oriented bipartite graph with score set A.

Proof. First assume $d > 2$. Induct on n. If n = 0, then by Theorem 2.1, there exists an oriented bipartite graph D(U, V) with score set A = {a}.

For n = 1, construct an oriented bipartite graph D(U, V) as follows.

Let U = $X_1 \cup X_2$, V = $Y_1 \cup Y_2$ with $X_1 \cap X_2 = \phi, Y_1 \cap Y_2 = \phi, |X_1| = |Y_1| = a, |X_2| = |Y_2| = ad - 2a > 0$ as $a > 0,\ d > 2$. Let every vertex of X_2 dominates each vertex of Y_1, and every vertex of Y_2 dominates each vertex of X_1, so that we get the oriented bipartite graph D(U, V) with

$$|U| = |X_1| + |X_2| = a + ad - 2a = ad - a,$$
$$|V| = |Y_1| + |Y_2| = a + ad - 2a = ad - a,$$

and the scores of vertices

$a_{x_1} = |V| + 0 - (ad - 2a) = ad - a - ad + 2a = a$, for all $x_1 \in X_1$,

$a_{x_2} = |V| + a - 0 = ad - a + a = ad$, for all $x_2 \in X_2$,

$a_{y_1} = |U| + 0 - (ad - 2a) = ad - a - ad + 2a = a$, for all $y_1 \in Y_1$,

and $a_{y_2} = |U| + a - 0 = ad - a + a = ad$, for all $y_2 \in Y_2$.

Thus, score set of D(U, V) is $A = \{a, ad\}$.

Assume the result to be true for all $p \geq 1$. We show that the result is true for p + 1.

Let a and d be positive integers with $a > 0$ and $d > 2$. Therefore, by induction hypothesis, there exists an oriented bipartite graph D(U, V) with

$$\mid U \mid=\mid V \mid= ad^p - (ad^{p-1} - ad^{p-2} + \ . \ . \ . \ (-1)^{p+1}a),$$

and $a, ad, ad^2, \ . \ . \ . \ , ad^p$ as the scores of the vertices of D(U, V). As $a > 0,\ d > 2$, therefore $ad^{p+1} - 2(ad^p - (ad^{p-1} - ad^{p-2} + \ . \ . \ . \ (-1)^{p+1}a)) > 0$. Now, construct an oriented bipartite graph $D(U_1,\ V_1)$ as follows.

Let $U_1 = U \cup X, V_1 = V \cup Y$ with $U \cap X = \phi, V \cap Y = \phi$,

$$\mid X \mid=\mid Y \mid= ad^{p+1} - 2(ad^p - (ad^{p-1} - ad^{p-2} + \ . \ . \ . \ (-1)^{p+1}a)).$$

Let every vertex of X dominates each vertex of V, and every vertex of Y dominates each vertex of U, so that we get the oriented bipartite graph $D(U_1, V_1)$ with
$|U_1| = |U| + |X| = |V| + |Y| = |V_1|$
$= ad^p - (ad^{p-1} - ad^{p-2} + \ . \ . \ . \ (-1)^{p+1}a) + ad^{p+1} - 2(ad^p - (ad^{p-1} - ad^{p-2} + \ . \ . \ . \ (-1)^{p+1}a))$
$= ad^{p+1} - (ad^p - (ad^{p-1} - ad^{p-2} + \ . \ . \ . \ (-1)^{p+1}a)),$
and since $|X| = |Y|$, therefore $a + |X| - |X| = a$, $ad + |X| - |X| = ad$, $ad^2 + |X| - |X| = ad^2, \ . \ . \ . \ , ad^p + |X| - |X| = ad^p$ are the scores of the vertices of U and V, and

$a_x = |V_1| + |V| - 0 = |U_1| + |U| - 0 = a_y = ad^{p+1} - (ad^p - (ad^{p-1} - ad^{p-2} + \ . \ . \ . \ (-1)^{p+1}a)) + ad^p - (ad^{p-1} - ad^{p-2} + \ . \ . \ . \ (-1)^{p-1}a) = ad^{p+1}$, for all $x \in X, y \in Y$.
Therefore, score set of $D(U_1, \ V_1)$ is $A = \{a, ad, ad^2, \ . \ . \ . \ , ad^p, ad^{p+1}\}$.

Now, assume d = 2. Then the set A becomes $A = \{a, 2a, 2^2a, \ . \ . \ . \ , 2^na\}$. Construct an oriented bipartite graph D(U, V) as follows.

Let

$$U = X_0 \cup X_1 \cup X_3 \cup X_4 \cup \ . \ . \ . \ \cup X_n,$$
$$V = Y_0 \cup Y_2 \cup Y_3 \cup Y_4 \cup \ . \ . \ . \ \cup Y_n$$

with $X_i \cap X_j = \phi$, $Y_i \cap Y_j = \phi$ $(i \neq j)$. Let $|X_0| = |X_1| = |Y_0| = |Y_2| = a$,
and for $3 \leq i \leq n$

$$|X_i| = |Y_i| = 2^ia - 2(\textstyle\sum_{j=0, j\neq 2}^{i-1} |X_j|), \tag{2.3.1}$$

which is clearly greater than zero. Let every vertex of X_i dominates each vertex of Y_j whenever $i > j$, and every vertex of Y_i dominates each vertex of X_j whenever $i > j$, so that we get the oriented bipartite graph D(U, V) with the scores of vertices

$a_{x_0} = |V| + 0 - \sum_{j=2}^{n} |Y_j| = \sum_{j=0, j\neq 1}^{n} |Y_j| - \sum_{j=2}^{n} |Y_j| = |Y_0| = a$, for all $x_0 \in X_0$,

$a_{x_1} = |V| + |Y_0| - \sum_{j=2}^{n} |Y_j| = \sum_{j=0, j\neq 1}^{n} |Y_j| + a - \sum_{j=2}^{n} |Y_j| = |Y_0| + a = 2a$, for all $x_1 \in X_1$,

$a_{y_0} = |U| + 0 - \sum_{j=1, j\neq 2}^{n} |X_j| = \sum_{j=0, j\neq 2}^{n} |X_j| - \sum_{j=1, j\neq 2}^{n} |X_j| = |X_0| = a$, for all $y_0 \in Y_0$,

$a_{y_2} = |U| + |X_0| + |X_1| - \sum_{j=3}^{n} |X_j| = \sum_{j=0, j\neq 2}^{n} |X_j| + a + a - \sum_{j=3}^{n} |X_j| = |X_0| + |X_1| + 2a = a + a + 2a = 4a$, for all $y_2 \in Y_2$,

and for $3 \leq i \leq n$

$a_{x_i} = |V| + \sum_{j=0, j\neq 1}^{i-1} |Y_j| - \sum_{j=i+1}^{n} |Y_j| = |U| + \sum_{j=0, j\neq 2}^{i-1} |X_j| - \sum_{j=i+1}^{n} |X_j| = a_{y_i} = \sum_{j=0, j\neq 2}^{n} \mid X_j \mid + \sum_{j=0, j\neq 2}^{i-1} \mid X_j \mid - \sum_{j=i+1}^{n} \mid X_j \mid = \sum_{j=0, j\neq 2}^{i} \mid X_j \mid + \sum_{j=0, j\neq 2}^{i-1} \mid X_j \mid = 2\sum_{j=0, j\neq 2}^{i-1} \mid X_j \mid + \mid X_i \mid = 2\sum_{j=0, j\neq 2}^{i-1} \mid X_j \mid + 2^ia - 2(\sum_{j=0, j\neq 2}^{i-1} \mid X_j \mid)$

(By equation (2.3.1))

$= 2^ia$, for all $x_i \in X_i, y_i \in Y_i$.
Therefore , score set of D(U, V) is $A = \{a, 2a, 2^2a, \ . \ . \ . \ , 2^na\}$.

The next result shows that every set of positive integers in arithmetic progression is a score set for some oriented bipartite graph.

Theorem 2.4 Let $A = \{a, a + d, a + 2d, \ . \ . \ . \ , a + nd\}$, where a and d are positive integers. Then, there exists an oriented bipartite graph with score set A.
Proof.(a). Let d > a so that $d - a > 0$. Construct an oriented bipartite graph D(U, V) as follows.

Let

$$U = X_0 \cup X_1 \cup \ . \ . \ . \ \cup X_n,$$
$$V = Y_0 \cup Y_1 \cup \ . \ . \ . \ \cup Y_n$$

with $X_i \cap X_j = \phi, Y_i \cap Y_j = \phi(i \neq j)$, and for $0 \leq i \leq n$

$$\mid X_i \mid = \mid Y_i \mid = \begin{cases} a, & if\ i\ is\ even, \\ d-a, & if\ i\ is\ odd. \end{cases} \tag{2.4.1}$$

Let every vertex of X_i dominates each vertex of Y_j whenever $i > j$, and every vertex of Y_i dominates each vertex of X_j whenever $i > j$, so that we get the oriented bipartite graph D(U, V) with

$$\mid U \mid = \sum_{i=0}^{n} \mid X_i \mid = \sum_{i=0}^{n} \mid Y_i \mid = \mid V \mid$$
$$= \begin{cases} a+d-a+a+d-a+ \ . \ . \ . \ +d-a+a, & if\ n\ is\ even, \\ a+d-a+a+d-a+ \ . \ . \ . \ +a+d-a, & if\ n\ is\ odd, \end{cases}$$
$$= \begin{cases} (\frac{n}{2}+1)a+\frac{n}{2}(d-a), & if\ n\ is\ even, \\ (\frac{n+1}{2})a+(\frac{n+1}{2})(d-a), & if\ n\ is\ odd, \end{cases}$$
$$= \begin{cases} \frac{nd}{2}+a, & if\ n\ is\ even, \\ (\frac{n+1}{2})d, & if\ n\ is\ odd, \end{cases} \tag{2.4.2}$$

and the scores of vertices

$a_{x_0} = |V| + 0 - \sum_{j=1}^{n} |Y_j| = |U| + 0 - \sum_{j=1}^{n} |X_j| = a_{y_0} = \sum_{j=0}^{n} |Y_j| - \sum_{j=1}^{n} = |Y_j| = \mid Y_0 \mid = a,$ for all $x_0 \in X_0,\ y_0 \in Y_0,$

and for $1 \leq i \leq n$

$a_{x_i} = |V| + \sum_{j=0}^{i-1} |Y_j| - \sum_{j=i+1}^{n} |Y_j| = |U| + \sum_{j=0}^{i-1} |X_j| - \sum_{j=i+1}^{n} |X_j| = a_{y_i} = \sum_{j=0}^{n} |X_j| + \sum_{j=0}^{i-1} |X_j| - \sum_{j=i+1}^{n} |X_j| = \sum_{j=0}^{i} |X_j| + \sum_{j=0}^{i-1} |X_j| = 2\sum_{j=0}^{i-1} |X_j| + |X_i|$

$$= \begin{cases} 2\sum_{j=0}^{i-1} |X_j|+a, & if\ i\ is\ even, \\ 2\sum_{j=0}^{i-1} |X_j|+d-a, & if\ i\ is\ odd, \end{cases} \quad \text{(By equation (2.4.1))}$$
$$= \begin{cases} 2(\frac{i-1+1}{2})d+a, & if\ i\ is\ even, \\ 2((\frac{i-1}{2})d+a)+d-a, & if\ i\ is\ odd, \end{cases} \quad \text{(By equation (2.4.2))}$$
$$= \begin{cases} a+id, & if\ i\ is\ even, \\ a+id, & if\ i\ is\ odd. \end{cases}$$

That is, $a_{x_i} = a_{y_i} = a + id$, for all $x_i \in X_i,\ y_i \in Y_i$ where $1 \leq i \leq n$. Therefore, score set of D(U, V) is A = {a, a + d, a + 2d , . . . , a + nd}.

(b). Let d = a. Then the set A becomes A = {a, 2a, 3a, . . . , (n + 1)a}. For n = 0, the result follows from Theorem 2.1 . Now, assume $n \geq 1$.

If n is odd, say $n = 2k - 1$ where $k \geq 1$, then construct an oriented bipartite graph D(U, V) as follows.

Let

$$U = X_0 \cup X_1 \cup X_3 \cup \ . \ . \ . \ \cup X_{2k-3} \cup X_{2k-1},$$
$$V = Y_0 \cup Y_2 \cup Y_4 \cup \ . \ . \ . \ \cup Y_{2k-2}$$

with $X_i \cap X_j = \phi, Y_i \cap Y_j = \phi(i \neq j),\ and\ |X_i| = |Y_j| = a,\ for\ all\ i \in \{0, 1, 3,\ . \ . \ . \ , 2k-1\}, j \in \{0, 2, 4,\ . \ . \ . \ , 2k-2\}$. Let every vertex of X_i dominates each vertex of Y_j whenever $i > j$, and every vertex of Y_i dominates each vertex of X_j whenever $i > j > 0$, so that we get the oriented bipartite graph D(U, V) with

$$| U |= \sum_{j\in\{0,1,3,\ \dots\ ,2k-1\}} | X_j |= a + (\tfrac{2k-1+1}{2})a = a + ka,$$
$$| V |= \sum_{j\in\{0,2,4,\ \dots\ ,2k-2\}} | Y_j |= a + (\tfrac{2k-2}{2})a = ka,$$

and the scores of vertices

$a_{x_0} =| V | +0 - 0 = ka$, for all $x_0 \in X_0$,

for $i \in \{1, 3,\ \dots\ , 2k-1\}$

$a_{x_i} = |V| + |Y_0| + \sum_{j\in\{2,4,\ \dots\ ,i-1\}} |Y_j| - \sum_{j\in\{i+1,i+3,\ \dots\ ,2k-2\}} |Y_j| = ka + a + (\frac{i-1}{2})a - (\frac{2k-2-(i-1)}{2})a = ka + a + ia - a - ka + a = (i+1)a$, for all $x_i \in X_i$,

$a_{y_0} = |U| + 0 - \sum_{j\in\{1,3,\ \dots\ ,2k-1\}} |X_j| = a + ka - (\frac{2k-1+1}{2})a = a$, for all $y_0 \in Y_0$,

and for $i \in \{2, 4,\ \dots\ , 2k-2\}$

$a_{y_i} = |U| + \sum_{j\in\{1,3,\ \dots\ ,i-1\}} |X_j| - \sum_{j\in\{i+1,i+3,\ \dots\ ,2k-1\}} |X_j| = a + ka + (\frac{i-1+1}{2})a - (\frac{2k-1+1-(i-1+1)}{2})a = a + ka + \frac{ia}{2} - ka + \frac{ia}{2} = (i+1)a, \quad for\ all\ y_i \in Y_i$.

Thus, score set of D(U, V) is $A = \{a, 2a, 3a,\ \dots\ , (2k-1)a, 2ka\}$.

Now, if n is even, say n = 2k where $k \geq 1$, then construct an oriented bipartite graph D(U, V) as follows.

Let

$$U = X_0 \cup X_1 \cup X_3 \cup\ \dots\ \cup X_{2k-1},$$
$$V = Y_0 \cup Y_2 \cup Y_4 \cup\ \dots\ \cup Y_{2k}$$

with $X_i \cap X_j = \phi, Y_i \cap Y_j = \phi\ \ (i \neq j)$, and $|X_i| = |Y_j| = a$, for all $i \in \{0, 1, 3,\ \dots\ , 2k-1\}$, $j \in \{0, 2, 4,\ \dots\ , 2k\}$. Let every vertex of X_i dominates each vertex of Y_j whenever $i > j$, and every vertex of Y_i dominates each vertex of X_j whenever $i > j > 0$, so that we get the oriented bipartite graph D(U, V) with (as in above) $|U| = a+ka$, $|V| = ka+a = a+ka$, and the scores of vertices

$a_{x_0} = ka + |Y_{2k}| = ka + a = (k+1)a$, for all $x_0 \in X_0$,

for $i \in \{1, 3,\ \dots\ , 2k-1\}$

$a_{x_i} = (i+1)a$, for all $x_i \in X_i$,

$a_{y_0} = a$, for all $y_0 \in Y_0$,

for $i \in \{2, 4,\ \dots\ , 2k-2\}$

$a_{y_i} = (i+1)a, \quad for\ all\ y_i \in Y_i$,

and $a_{y_{2k}} = |U| + \sum_{j\in\{1,3,\ \dots\ ,2k-1\}} |X_j| - 0 = a + ka + (\frac{2k-1+1}{2})a = (2k+1)a$, for all $y_{2k} \in Y_{2k}$.

Thus, score set of D(U, V) is $A = \{a, 2a, 3a,\ \dots\ , 2ka, (2k+1)a\}$.

(c). Let $d < a$ so that $a - d > 0$. For n = 0 or 1, the result follows from Theorem 2.1 . Now, assume that $n \geq 2$.

If n is even, say n = 2k where $k \geq 1$, then construct an oriented bipartite graph D(U, V) as follows.

Let

$$U = X_0 \cup X_1 \cup X_3 \cup\ \dots\ \cup X_{2k-1},$$
$$V = Y_0 \cup Y_2 \cup Y_4 \cup\ \dots\ \cup Y_{2k}$$

with $X_i \cap X_j = \phi$, $Y_i \cap Y_j = \phi\ \ (i \neq j), |X_0| = |Y_0| = a$, and $|X_i| = |Y_j| = d$, for all $i \in \{1, 3,\ \dots\ , 2k-1\}$, $j \in \{2, 4,\ \dots\ , 2k\}$. Let every vertex of X_i dominates each vertex of Y_j whenever $i > j > 1$, every vertex of X_i dominates d vertices of Y_0 (out of a) whenever $i > 2$, and every vertex of Y_i dominates each vertex of X_j whenever $i > j > 0$, so that we get the oriented bipartite graph D(U, V) with

$$| U |= \sum_{j\in\{0,1,3,\ .\ .\ .\ ,2k-1\}} | X_j |= a + (\tfrac{2k-1+1}{2})d = a + kd,$$
$$| V |= \sum_{j\in\{0,2,4,\ .\ .\ .\ ,2k\}} | Y_j |= a + (\tfrac{2k}{2})d = a + kd,$$

and the scores of vertices

$a_{x_0} =| V | +0 - 0 = a + kd$, for all $x_0 \in X_0$,

$a_{x_1} =| V | +0 - \sum_{j\in\{2,4,\ .\ .\ .\ ,2k\}} | Y_j |= a + kd - (\frac{2k}{2})d = a$, for all $x_1 \in X_1$,

for $i \in \{3, 5,\ .\ .\ .\ , 2k - 1\}$

$a_{x_i} = |V| + d + \sum_{j\in\{2,4,\ .\ .\ .\ ,i-1\}} |Y_j| - \sum_{j\in\{i+1,i+3,\ .\ .\ .\ ,2k\}} |Y_j| = a + kd + d + (\frac{i-1}{2})d - (\frac{2k-(i-1)}{2})d = a + kd + d + (i - 1)d - kd = a + id, \quad for\ all\ x_i \in X_i,$

$a_{y_0} = |U| + 0 - 0 = a + kd$, for the $a - d$ vertices of Y_0,

$a_{y'_0} =| U | +0-\sum_{j\in\{3,5,\ .\ .\ .\ ,2k-1\}} |X_j| = a+kd-(\frac{2k-1+1-(1+1)}{2})d = a+kd-kd+d = a+d$, for the remaining d vertices of Y_0,

and for $i \in \{2, 4,\ .\ .\ .\ , 2k\}$

$a_{y_i} = |U| + \sum_{j\in\{1,3,\ .\ .\ .\ ,i-1\}} |X_j| - \sum_{j\in\{i+1,i+3,\ .\ .\ .\ ,2k-1\}} |X_j| = a + kd + (\frac{i-1+1}{2})d - (\frac{2k-1+1-(i-1+1)}{2})d = a + kd + \frac{id}{2} - kd + \frac{id}{2} = a + id$, for all $y_i \in Y_i$.

Therefore, score set of D(U, V) is $A = \{a, a + d, a + 2d,\ .\ .\ .\ , a + (2k - 1)d, a + 2kd\}$.

Now, if n is odd, say n = 2k + 1 where $k \geq 1$, then construct an oriented bipartite graph D(U, V) as follows.

Let

$$U = X_0 \cup X_1 \cup X_3 \cup\ .\ .\ .\ \cup X_{2k-1} \cup X_{2k+1},$$
$$V = Y_0 \cup Y_2 \cup Y_4 \cup\ .\ .\ .\ \cup Y_{2k}$$

with $X_i \cap X_j = \phi, Y_i \cap Y_j = \phi\ (i \neq j), |X_0| = |Y_0| = a$, and $|X_i| = |Y_j| = d$, for all $i \in \{1, 3,\ .\ .\ .\ , 2k + 1\}, j \in \{2, 4,\ .\ .\ ., 2k\}$. Let every vertex of X_i dominates each vertex of Y_j whenever $i > j > 1$, every vertex of X_i dominates d vertices of Y_0 (out of a) whenever $i > 2$, and every vertex of Y_i dominates each vertex of X_j whenever $i > j > 0$, so that we get the oriented bipartite graph D(U, V) with (as in above) $|U| = a+kd+d = a+(k+1)d, |V| = a+kd$, and the scores of vertices

$a_{x_0} = a + kd$, for all $x_0 \in X_0$,

$a_{x_1} = a$, for all $x_1 \in X_1$,

for $i \in \{3, 5,\ .\ .\ .\ , 2k - 1\}$

$a_{x_i} = a + id$, for all $x_i \in X_i$,

$a_{x_{2k+1}} = |V| + d + \sum_{j\in\{2,4,\ .\ .\ .\ ,2k\}} |Y_j| - 0 = a + kd + d + (\frac{2k}{2})d = a + (2k + 1)d$, for all $x_{2k+1} \in X_{2k+1}$,

$a_{y_0} = a + kd + |X_{2k+1}| = a + kd + d = a + (k + 1)d$, for the $a - d$ vertices of Y_0,

$a_{y'_0} = a + d$, for the remaining d vertices of Y_0,

and for $i \in \{2, 4,\ .\ .\ .\ , 2k\}$

$a_{y_i} = a + id$, for all $y_i \in Y_i$.

Hence, score set of D(U, V) is $A = \{a, a + d, a + 2d,\ .\ .\ .\ , a + 2kd, a + (2k + 1)d\}$, and the proof is complete.

Remark. We note that Theorems 2.1, 2.2, and 2.4 cannot be extended to state that any set of nonnegative integers A is a score set of some oriented bipartite graph when $|A|$ = 1, 2, 3, or when A is an arithmetic progression, for instance, there is no oriented bipartite graph with score set {0}, {0, 1}, or {0, 1, 2}.

We conclude with the following conjecture.

Conjecture. Every finite set of positive integers is a score set for some oriented bipartite graph.

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